An effective version of Nadkarni's thm (w/ Kechris)

Def$^{\text{n}}$ A countable Borel eq. relation (CBER) E on sbs X is an eq. relation s.t. $E \subseteq X^2$ is Borel & all E-classes are ctbl.

A Borel prob. measure μ on X is E-inv. if $\mu(A) = \mu(B)$ whenever $\exists f: A \to B$ partial Borel bij. st $x E f(x)\ \forall x \in A$.

μ is E-ergodic if $\mu(A) \in \{0,1\}$ $\forall E$-inv. A.

Eg E_0 on $2^{\mathbb{N}} = \mathcal{C}$

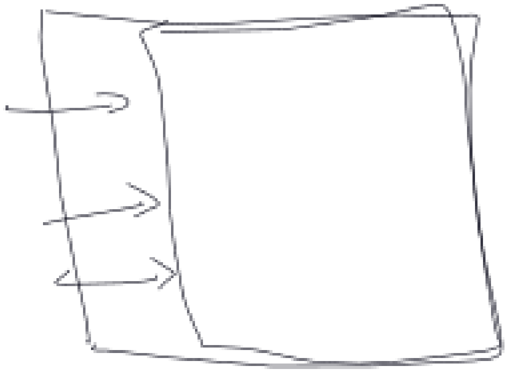
$x E_0 y$ iff $\exists N \forall n > N \ x(n) = y(n)$

$(1/2, 1/2)^{\mathbb{N}}$ is E_0-inv. & erg.

$\Gamma \curvearrowright 2^{\Gamma}$ E_{Γ} orbit eq. rel.

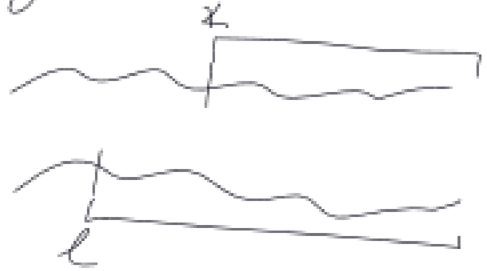
$(\alpha, 1-\alpha)^{\Gamma}$ is Γ-inv. & erg. for $\alpha \in (0,1)$

$$[A]_E = \{x : \exists y \in A (x E y)\}$$



E_t on $\mathcal{C}$

$x E_t y$ iff $\exists k, l \forall n, \ x(k+n) = y(l+n)$



$\underline{x E_t f(x)}$

Spse μ is an $\underline{E_t\text{-inv. meas.}}$

let $f : \mathcal{C} \to \mathcal{C}$ take x to $0^\frown x$

$f : \mathcal{C} \hookrightarrow\!\!\!\!\to \{0^\frown x : x \in \mathcal{C}\}$

$1 = \mu(\mathcal{C}) = \mu(\{0^\frown x : x \in \mathcal{C}\}) = 1$

$0 = \mu(\{1^\frown x : x \in \mathcal{C}\}) = \mu([\{1^\frown x : x \in \mathcal{C}\}]_{E_t})$

$= \mu(\mathcal{C}) = 1$ ↯

Def^{n} A CBER E on X is __compressible__ if it has a Borel __compression__ i.e. an inj. Borel map $f: X \to X$ s.t. $\forall x,\ x E f(x)$ & $f(C) \subsetneq C$ $\forall$ E-classes C

$[X \setminus f(X)]_E = X$

__Obs__ E compressible $\Rightarrow$ no E-inv. meas.

__Thm__ (Nadkarni '90, Becker-Kechris) Let E be a CBER. Then exactly one of the following hold:

(a) E is compressible

(b) E admits an inv. Borel prob. meas.

Thm (Ergodic decomposition thm, Farrell, Varadarajan '60s)

Let E be a Δ^1_1 CBER on X, $INV_E \neq \emptyset$. Then $EINV_E \neq \emptyset$ $\exists Z$ Δ^1_1 w/ a Δ^1_1 comp.

and $\exists \pi : X \setminus Z \to P(X)$ s.t. prob meas. on X

Δ^1_1

(i) π is an E-inv. Borel surjection onto $EINV_E$

(ii) $e(\pi^{-1}(e)) = 1$ & e is the unique E-erg inv. meas. on $\pi^{-1}(e)$ for $e \in EINV_E$

(iii) $\forall \mu \in INV_E$, $\mu = \int \pi(x) \, d\mu(x)$

$$\mu(A) = \int \pi(x)(A) \, d\mu(x)$$

$[0,1]^{\mathbb{N}^{<\mathbb{N}}}$

$\mu(\emptyset) = 1$ $\quad \mu(s) = \sum_n \mu(s^\frown n)$

$e(s) = \mu(N_s)$

Effective descriptive set theory

$X = \mathcal{N} = \mathbb{N}^{\mathbb{N}}$ $\quad N_s = \{x : s \subseteq x\}$

$s \in \mathbb{N}^{<\mathbb{N}}$

~~arbitrary~~ computable unions of N_s

Σ^0_1 $\quad \Sigma^0_2$

Π^0_1 $\quad \Pi^0_2$

branches through ~~arbitrary~~ computable trees

tree on $\mathbb{N}$

. . . ω_1^{CK}

Souslin-Kleene

~~Borel~~ HYP $= \Delta^1_1$

arbitrary ctbl unions/intersections

Σ^1_1 $\quad \exists y A(x,y)$ $\quad \Pi^0_1$

$\bigcup_y A^y$

Π^1_1

. . . "uniform" continuum sized unions of ~~Borel~~ HYP Π^0_1 sets

$U = \bigcup_{s \in A} N_s$

computable

Def^n $\alpha \in \mathcal{N}$ is Δ^1_1 if $\{\alpha\} \in \Delta^1_1$

Thm (Harrison, '67) Let $A \in \Sigma^1_1$. Then exactly one holds:

(a) $A \subseteq \Delta^1_1$

(b) $\exists$ cont. embedding $f: \mathcal{C} \to A$

"Relativization"

Eg $\boldsymbol{\Sigma}^1_1 = \bigcup_{\substack{A \\ \text{oracles}}} \Sigma^1_1(A)$

Eg "$\exists \alpha \in \Delta^1_1$" is Π^1_1 $\rightsquigarrow$ Lusin-Novikov, "$\exists !$" is Π^1_1

"$\cancel{\mu}$ is μ-hyp"

[FKSV]

Q Is there an effective witness to compressibility?

A Yes!

Thm E Δ^1_1 CBER on $\mathcal{N}$. either E has a Δ^1_1 compression
or admits an inv. Borel prob. meas

Cor eff. erg. decomp thm.

$$\underset{\sim}{\Sigma}^0_1 = \bigcup_{\alpha} \Sigma^0_1(\alpha)$$

$\underset{\sim}{\Pi}^0_1$